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Elastic behaviour of an orthotropic beam/one-dimensional plate of uniform and variable thickness

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Abstract. In this paper, the analysis of the title problem is based on mixed first-order thick-beam one-dimensional plate theory, and on using a small-parameter approach towards its numerical solution. The boundary conditions at the edges of the beam may be quite general, and between these two edges the beam may have varying thickness. Closed-form solutions have been developed for the static response of orthotropic beams with nonlinear thickness variation subjected to uniform loading. The accuracy of the present model is demonstrated by problems for which exact solutions and numerical results are available, and the results are also presented for a variety of problems whose solutions are not available in the literature.

Key words: bending response, orthotropic beams, small-parameter method, variable thickness.

1. Introduction

Current research activities on the static response of beams focus invariably on various complicating effects, such as shear deformation, general polygonal boundaries, material anisotropy, loading, thickness non-uniformity, and so on. Simplifying assumptions as to stress-distribution states, namely that certain stresses are zero or constant in the domain of interest, are made. This decreases the number of dependent variables to be dealt with and permits the solution of several problems of interest. In the structural studies, the assumptions are made primarily about displacement fields. Also, they usually concern certain aspects of the constitutive law to be employed. The variational process will then give us the proper equilibrium equations and the appropriate boundary conditions for the problems.

Most important, if appropriate assumptions for the displacement field are made, the number of variables is significantly reduced and this facilitates the actual computations. For the classical beam theory (CBT), the problem is reduced to a single dependent variable $w(x)$ representing the deflection of the centerline of the beam. It is clear from the displacement field of this theory that all strains, except ε_{xx}, are zero. Also, the bending deformation is given in terms of the deformation of the centerline of the beam. So, the CBT does not include the effects of shear deformation. For short stubby beams, this contribution obviously cannot be neglected. For this reason, the first-order beam theory (FBT) is presented as a means of accounting for the effects of shear in a simple manner.

Variable-thickness beams are widely used in many kinds of high-performance surface and air vehicles. This requires further refining of mathematical models describing their behaviour. The selection of a model that adequately describes the response of a structure is made on the basis of the relative beam thickness and type of loading.

When the thickness along the length of a beam is variable, its closed-form solution becomes very complex, even for simple cases. For uniform-thickness beams, however, many solutions for elastic analysis have been developed and are available in the literature [1–7]. The bending characteristics of thin and thick beams of uniform thickness may be obtained by simply specifying the beam aspect (length-to-thickness) ratio and, in addition, Poisson's ratio, if distributed loads are involved. The bending characteristics of such beams have been studied at great length [8–14].

The analysis of elastic structures of variable thickness is very rare in the literature [15–18]. In this paper, the model of a thick beam that is also applicable to a 1-D plate, is put forward. The complicating effects considered herein concern variable thickness, various boundary conditions and material variability in that the beams could be orthotropic or, especially, isotropic. The bending problem of an orthotropic beam with nonlinear thickness variation is investigated. Various cases of thickness variation and boundary conditions are studied. With the help of a small-parameter method, a wide variety of deflections, bending moments and normal stresses is presented. Several numerical examples, including comparisons with some results available for uniform beams, are given to illustrate the salient features of the present formulation. In addition, appropriate conclusions concerning the thickness variation, thickness parameter, and various effects related to boundary conditions are formulated.

2. Governing equations

Consider a rectangular beam of length, L, width b, made of an orthotropic material having its coordinate axes (x, y, z) chosen such that x is the axial coordinate. Let the thickness h of the beam be varying in the x direction only, that is, $h = h(x)$. All applied loads and geometry are such that the displacements (u_1, u_2, u_3) along the coordinates (x, y, z) are functions of the x- and z-coordinates only. Here we further assume that the displacement u_2 is identically zero. Shown in Figure 1 is the profile of a beam of variable thickness with uniform, linear, quadratic, and cubic thickness variations.

The beam surfaces are assumed to be subjected to the following traction field:

$$\hat{t}\,(x, y, 0) = \left(0, 0, -\frac{q(x)}{b}\right), \quad \hat{t}(x, y, h) = (0, 0, 0). \tag{1}$$

In what follows we have substituted σ_1, σ_3, and σ_5 for the conventional σ_{xx}, σ_{zz}, and σ_{zx}, respectively. Thus, the boundary conditions on the beam surfaces are:

$$\sigma_3 = -\frac{q(x)}{b}, \quad \sigma_5 = 0 \quad \text{at} \quad z = 0; \quad \sigma_3 = 0, \quad \sigma_5 = 0 \quad \text{at} \quad z = h. \tag{2}$$

The displacement-based one-dimensional field is assumed to be of the form

$$u_1(x, z) = -z\psi(x) = -z\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \varphi(x)\right), \quad u_3(x, z) = w(x), \tag{3}$$

where the function $w(x)$ is the transverse normal component (deflection) of the displacement of points on the neutral axis of the beam and $\psi(x)$ is the rotation of line elements along the centerline due to bending only. Note that the shear strain is the same at all points over a given cross-section of the beam. Thus, the angle $\varphi(x)$, used hitherto for rotation of elements along the centerline, is used to measure the shear angle at all points in the cross-section of the beam at position x. The strain-displacement relations are now as follows:

$$\varepsilon_1 = \varepsilon_{xx} = -z\frac{\mathrm{d}\psi}{\mathrm{d}x}, \quad \varepsilon_3 = \varepsilon_{zz} = 0, \quad \varepsilon_5 = 2\varepsilon_{xz} = \varphi(x). \tag{4}$$

The displacement field of classical beam theory (CBT) can be obtained from the present first-order beam theory (FBT) by setting $\varphi = 0$.

The present theory of beam behavior will be derived by application of the mixed variational formula (see, *e.g.*, [19, 20]),

$$0 = \int_{t_2}^{t_1} \left\{ \iiint_V \left[\rho \ddot{u}_i \delta u_i + \delta\left(\sigma_{ij}\varepsilon_{ij}\right) - \delta R \right] \mathrm{d}v + \delta\Pi \right\} \mathrm{d}t, \tag{5}$$

where ρ is the density of the undeformed body, (t_1, t_2) is a time interval, and R is the complementary energy density. The potential energy Π of the applied loads can be defined as a function of the displacement field u_i and the applied loads as follows:

$$\Pi = -\iiint_V B_i u_i \mathrm{d}v - \iint_{S_\sigma} F_i u_i \mathrm{d}s - \iint_{S_u} n_j \sigma_{ij} \left(u_i - u_i^*\right) \mathrm{d}s, \tag{6}$$

where n_j are the components of the unit vector along the outward normal to the total surface $S_\sigma + S_u$; B_i are the body forces measured per unit volume of the undeformed body; F_i are the prescribed components of the stress vector, per unit area of the surface S_σ, and u_i^* are the prescribed components of the displacements of the remaining surface S_u. In the absence of body forces and prescribed displacements, we have for the first variation of Π,

$$\delta\Pi = -\int_{-L/2}^{+L/2} q(x)\delta w \, \mathrm{d}x. \tag{7}$$

It is clear that both the displacements and the stress are taken to be arbitrary in the mixed variational formula (5). Then, the non-vanishing stresses are assumed to be of the form (for more details, we refer to [10, 20]):

$$\sigma_1 = \left(z - \frac{h}{2}\right) G_1(x), \quad \sigma_3 = \sum_{r=0}^{3} z^r G_3^{(r)}(x), \quad \sigma_5 = \frac{z}{h}\left(1 - \frac{z}{h}\right) G_5(x). \tag{8}$$

The functions G_1 and G_5 may be easily obtained from the observation that the in-plane normal stress σ_1 and the transverse shear stress σ_5 satisfy the following stress resultants:

$$M = b\int_0^h z\sigma_1 \, \mathrm{d}z, \qquad Q = b\int_0^h \sigma_5 \, \mathrm{d}z, \tag{9-a}$$

where M is the bending moment and Q the shear force. Also, the functions $G_3^{(r)}$ arise from the fact that the transverse normal stress σ_3 satisfies the boundary conditions (2), as well as the following conditions

$$\int_0^h \sigma_3 \, \mathrm{d}z = 0, \qquad \int_0^h z\sigma_3 \, \mathrm{d}z = 0. \tag{9-b}$$

In this case, the stresses (8) can be written in the following final form:

$$\sigma_1 = \left(z - \frac{h}{2}\right)\frac{M}{I}, \quad \sigma_3 = -\frac{q}{b}\left[1 - 8\left(\frac{z}{h}\right) + 10\left(\frac{z}{h}\right)^2\right]\left(1 - \frac{z}{h}\right), \quad \sigma_5 = \frac{6Q}{A}\frac{z}{h}\left(1 - \frac{z}{h}\right). \tag{10}$$

where $I = bh^3/12$ and $A = bh$. It should be noted that the transverse shear stress σ_5 is a function of z and satisfies the boundary condition (2); it vanishes on the bounding planes ($z = 0$ and $z = h$). For the present theory the first variation of the complementary energy density takes the form [21, pp. 24–37]:

$$\delta R = (a_{11}\sigma_1 + a_{13}\sigma_3)\,\delta\sigma_1 + a55\sigma_5\delta\sigma_5, \tag{11}$$

where the elastic constants a_{ij} can be expressed in terms of the engineering orthotropic characteristics as

$$a_{11} = \frac{1}{E_1}, \qquad a_{13} = -\frac{\nu_{13}}{E_1}, \qquad a_{55} = \frac{1}{G_{13}}, \tag{12}$$

in which E_1, G_{13} and ν_{13} stand for Young's modulus, shear modulus and Poisson's ratio, respectively. Note that we have for an isotropic beam

$$a_{11} = \frac{1}{E}, \qquad a_{13} = -\frac{\nu}{E}, \qquad a_{55} = \frac{1}{G} = \frac{2(1+\nu)}{E}. \tag{13}$$

The next step in deriving the governing equations consists of substituting (3), (4), (7), (10) and (11) in the variational formula (5). The extremum conditions give the following equilibrium equations:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = -q, \qquad \frac{\mathrm{d}M}{\mathrm{d}x} - Q = 0. \tag{14}$$

The mixed variational formula (5) gives also the natural boundary conditions for this problem. They are given after restating all of them in terms of force and moment as follows (at $x = \mp L/2$):

1. Either ψ is specified or $M = 0$,
2. Either w is specified or $Q = 0$. (15)

Now, assuming that the beam has variable thickness h in the x-direction, we can write

$$h = h_0[1 + \lambda f_n(x)], \tag{16}$$

where h_0 is the constant reference thickness value located as shown in Figure 1; $f_n(x)$ describes the thickness variation in which n is the degree of non-uniformity and λ is a small parameter. The values of λ can be chosen so that the function $f_n(x)$ can be constrained to have values satisfying

$$|f_n(x)| \le 1. \tag{17}$$

In addition to the equilibrium equations and boundary conditions, the mixed variational formula gives also the stress resultants as follows:

$$M = -c_{11}\left[1 + \lambda f_n(x)\right]^3 \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{\mathrm{d}\varphi}{\mathrm{d}x}\right), \qquad Q = c_{55}\left[1 + \lambda f_n(x)\right]\varphi, \tag{18}$$

where

$$c_{11} = \frac{bE_1h_0^3}{12}, \qquad c_{55} = \frac{5}{6}bh_0G_{13}. \tag{19}$$

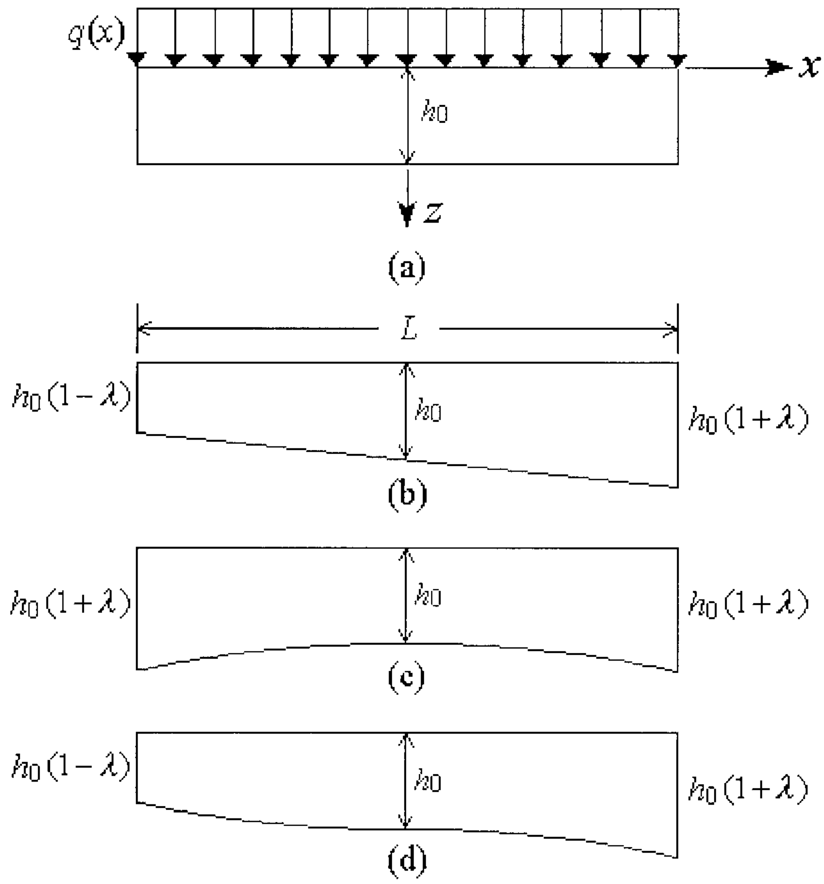

Figure 1. Variable thickness beam-one-dimensional plate: (a) uniform thickness beam subjected to uniformaly distributed load, (b) linear thickness variation beam, (c) quadratic thickness variation beam, and (d) cubic thickness variation beam.

For the elastic case it can be proved that a variable-thickness beam represented by (14) may be replaced by a set of beams of uniform thickness h_0. Substitution of (18) in (14) results in the following linear differential equations for $w(x)$ and $\varphi(x)$,

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - \frac{\mathrm{d}^3 \varphi}{\mathrm{d}x^3} = \frac{q}{c_{11}}\left[1+\lambda f_n(x)\right]^{-3} - 6\lambda\left[1+\lambda f_n(x)\right]^{-1}\frac{\mathrm{d}f_n}{\mathrm{d}x}\left(\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} - \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}\right) \tag{20}$$

$$-3\lambda\left[1+\lambda f_n(x)\right]^{-1}\frac{\mathrm{d}^2 f_n}{\mathrm{d}x^2}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{\mathrm{d}\varphi}{\mathrm{d}x}\right) - 6\lambda^2\left[1+\lambda f_n(x)\right]^{-2}\left(\frac{\mathrm{d}f_n}{\mathrm{d}x}\right)^2\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{\mathrm{d}\varphi}{\mathrm{d}x}\right),$$

$$\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} - \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = -\frac{c_{55}}{c_{11}}\left[1+\lambda f_n(x)\right]^{-2}\varphi - 3\lambda\left[1+\lambda f_n(x)\right]^{-1}\frac{\mathrm{d}f_n}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{\mathrm{d}\varphi}{\mathrm{d}x}\right). \tag{21}$$

The appropriate solution to the linear differential equations (20) and (21) can be written with the help of the small-parameters method as

$$w(x) = L\sum_{s=0}^{\infty}\lambda^s w_s(x), \qquad \varphi(x) = \sum_{s=0}^{\infty}\lambda^s \varphi_s(x), \tag{22}$$

where the parameter λ is considered to be smaller than or equal to 0·5.

Note that w_s and φ_s represent series of solutions corresponding to the series λ_s appearing on the right side of the equations. Equating the coefficients of $\lambda^s (s = 0, 1, 2, \ldots)$ for both sides of (20) and (21), we can easily get two sets of differential equations representing equivalent flat beams of uniform thickness h_0. The solution of each differential equation may be carried out by using suitable known methods of analysis for flat beams. Rigorous as well as numerical methods of analysis may be used for this purpose. The solution applies to all boundary conditions and all continuous thickness variations with continuous first and second derivatives.

Introducing the dimensionless variable $\xi (= x/L)$ and its differential operator $\mathrm{D}(= \mathrm{d}/\mathrm{d}\xi)$, then substituting (22) in (20) and (21), we can write

$$\mathrm{D}^4 w_m - \mathrm{D}^3 \varphi_m = g_m(\xi), \qquad m = 0, 1, 2, \ldots \tag{23}$$

$$\mathrm{D}^3 w_m - \mathrm{D}^2 \varphi_m + c\varphi_m = p_m(\xi), \qquad m = 0, 1, 2, \ldots \tag{24}$$

where $c = c_{55} L^2 / c_{11}$. The functions $g_m(\xi)$ and $p_m(\xi)$ for $m = 0, 1, 2, \ldots$ are given by:

$$\begin{aligned}
g_0 &= \frac{qL^3}{c_{11}}, \\
g_1 &= -3 f_n g_0 - 6\mathrm{D} f_n \mathrm{D}^2 (\mathrm{D}w_0 - \varphi_0) - 3\mathrm{D}^2 f_n \mathrm{D} (\mathrm{D}w_0 - \varphi_0), \\
g_2 &= 6 f_n^2 g_0 - 6\mathrm{D} f_n \left[\mathrm{D}^2 (\mathrm{D}w_1 - \varphi_1) - f_n \mathrm{D}^2 (\mathrm{D}w_0 - \varphi_0)\right] \\
&\quad -3\mathrm{D}^2 f_n \left[\mathrm{D} (\mathrm{D}w_1 - \varphi_1) - f_n \mathrm{D} (\mathrm{D}w_0 - \varphi_0)\right] - 6 (\mathrm{D} f_n)^2 \,\mathrm{d} (\mathrm{D}w_0 - \varphi_0), \\
g_3 &= -10 f_n^3 g_0 - 6\mathrm{D} f_n \left[\mathrm{D}^2 (\mathrm{D}w_2 - \varphi_2) - f_n \mathrm{D}^2 (\mathrm{D}w_1 - \varphi_1) + f_n^2 \mathrm{D}^2 (\mathrm{D}w_0 - \varphi_0)\right] \\
&\quad -3\mathrm{D}^2 f_n \left[\mathrm{D} (\mathrm{D}w_2 - \varphi_2) - f_n \mathrm{D} (\mathrm{D}w_1 - \varphi_1) + f_n^2 \mathrm{D} (\mathrm{D}w_0 - \varphi_0)\right] \\
&\quad -6 (\mathrm{D} f_n)^2 \left[\mathrm{D} (\mathrm{D}w_1 - \varphi_1) - 2 f_n \mathrm{D} (\mathrm{D}w_0 - \varphi_0)\right], \\
&\vdots \\
p_0 &= 0, \\
p_1 &= 2c f_n \varphi_0 - 3\mathrm{D} f_n \mathrm{D} (\mathrm{D}w_0 - \varphi_0), \\
p_2 &= c f_n (2\varphi_1 - 3 f_n \varphi_0) - 3\mathrm{D} f_n \left[\mathrm{D} (\mathrm{D}w_1 - \varphi_1) - f_n \mathrm{D} (\mathrm{D}w_0 - \varphi_0)\right], \\
p_3 &= c f_n \left(2\varphi_n - 3 f_n \varphi_1 + 4 f_n^2 \varphi_0\right) - 3\mathrm{D} f_n \left[\mathrm{D} (\mathrm{D}w_2 - \varphi_2) - f_n \mathrm{D} (\mathrm{D}w_1 - \varphi_1)\right. \\
&\quad \left. + f_n^2 \mathrm{D} (\mathrm{D}w_0 - \varphi_0)\right], \\
&\vdots
\end{aligned}$$

For practical applications, we may obtain an accurate solution by using only the first two or three equations form the above sets of equations.

3. Exact solution for bending

The first of the preceding sets of Equations (23) and (24), namely for $m = 0$, represent a flat beam with constant loading g_0 that is identical to the load applied on the original variable thickness beam. The remaining equations in these sets represent flat beams with different loads $g_1, g_2, g_3, \ldots$ and so on. These loads can be determined once the displacements from the preceding equations have been determined. Thus, Equations (23) and (24) are sets of equations describing an equivalent system of flat beams that replace the original variable-thickness beam.

Now, differentiating (24) with respect to ξ and using (23) in the resulting equation, we get:

$$c\mathrm{D}\varphi_m = \mathrm{D}p_m(\xi) - g_m(\xi), \qquad m = 0, 1, 2, \ldots \tag{25}$$

The general solution of the above differential equation is written as

$$\varphi_m = (C_0)_m + \frac{1}{c}p_m(\xi) - \frac{1}{c}\mathrm{D}^{-1}[g_m(\xi)], \qquad m = 0, 1, 2, \ldots \tag{26}$$

Using the above relation in (24), we then get, on rearranging terms, the desired equation for the deflection w:

$$\mathrm{D}^3 w_m = -c\,(C_0)_m + \frac{1}{c}\mathrm{D}^2 p_m(\xi) - \frac{1}{c}\mathrm{D}g_m(\xi) + \mathrm{D}^{-1}[g_m(\xi)], \quad m = 0, 1, 2, \ldots \tag{27}$$

The general solution to the above differential equation may be given as follows:

$$w_m = (C_1)_m + (C_2)_m\,\xi + (C_3)_m\,\xi^2 + k_m(\xi), \quad m = 0, 1, 2, \ldots \tag{28}$$

where $k_m(\xi)$ is the particular solution of (27),

$$k_m(\xi) = -\frac{c}{6}\,(C_0)_m\,\xi^3 + \frac{1}{c}\mathrm{D}^{-1}[p_m(\xi)] - \frac{1}{c}\mathrm{D}^{-2}[g_m(\xi)] + \mathrm{D}^{-4}[g_m(\xi)]. \tag{29}$$

Note that, in the above Equations (26), (27) and (29), the integral operator D^{-p} is used,

$$\mathrm{D}^{-p}[\;] = \underbrace{\int \;\ldots\; \int}_{p \text{ times}} [\;]\mathrm{d}\xi \ldots \mathrm{d}\xi. \tag{30}$$

The boundary conditions for hinged (H), clamped (C), and free (F) at the edges $x = \mp L/2$ (or $\xi = \mp 1/2$) will be written as:

$$\begin{aligned} H &: \; w = M = 0, \\ C &: \; w = \psi = 0, \\ F &: \; M = Q : 0. \end{aligned} \tag{31}$$

By following the present theory of beams, the bending moment M and the shear force Q may be expressed in terms of the displacements w and φ in the z-direction of the beam by substituting (22) in (18). Also, the normal stress σ_1 through the thickness of the beam $(\eta = z/h)$ may be expressed in terms of w and φ as follows:

$$\sigma_1 = -\frac{12q}{b}\left(\frac{L}{h_0}\right)^2\left(\eta - \frac{1}{2}\right)\left[1 + \lambda f_n(x)\right]\sum_{s=0}^{\infty}\lambda^s \mathrm{D}\psi_s. \tag{32}$$

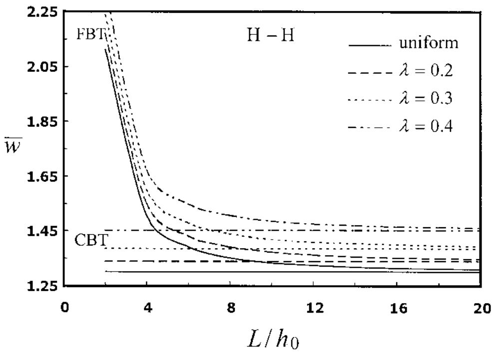

Figure 2. Dimensionless deflection ($\overline{w}$) *vs.* L/h_0 ratio at the center of a hinged-hinged isotropic beam with linear thickness variation ($n = 1$).

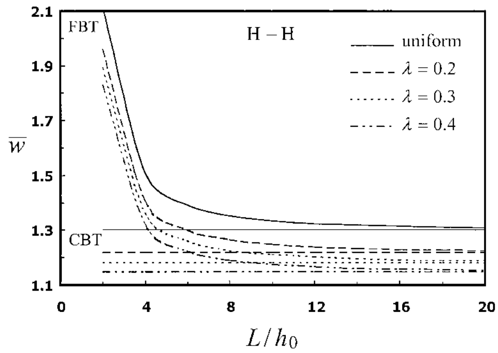

Figure 3. Dimensionless deflection ($\overline{w}$) *vs.* L/h_0 ratio at the center of a hinged-hinged isotropic beam with quadratic thickness variation ($n = 2$).

For the sake of comparison the counterparts of (23) and (24), derived as per CBT is also

$$\mathrm{D}^4 w_m = \overline{g}_m(\xi), \qquad m = 0, 1, 2, \ldots \tag{33}$$

In this case, the functions $\overline{g}_m(\xi)$ for $m = 0, 1, 2, \ldots$ follow directly from the corresponding functions $g_m(\xi)$ by setting $\varphi_m = 0$. The general solution of (33) is given as follows:

$$w_m = \left(\overline{C}_0\right)_m + \left(\overline{C}_1\right)_m \xi + \left(\overline{C}_2\right)_m \xi^2 + \left(\overline{C}_3\right)_m \xi^3 + \overline{k}_m(\xi), \quad m = 0, 1, 2, \ldots \tag{34}$$

where $\overline{k}_m(\xi)$ is the particular solution of (33) which is given by

$$\overline{k}_m(\xi) = \mathrm{D}^{-4}[\overline{g}_m(\xi)]. \tag{35}$$

As concerns the boundary conditions at $x = \mp L/2$ associated with the CBT ($\psi = -\mathrm{d}w/\mathrm{d}x$), these are similar to those given by (31) for FBT with the single exception of free-edge conditions, where $(31)_3$ is to be replaced by:

$$F: \quad M = \frac{\mathrm{d}M}{\mathrm{d}x} = 0. \tag{36}$$

The coefficients $(C_i)_m$ and $\left(\overline{C}_i\right)_m$ result from applying the boundary conditions (31) and (36) for FBT and CBT, respectively.

As a special case, the solution to the present thick-beam problem is given in the literature by superposing the solution for a hinged-hinged isotropic flat beam under a uniform load of intensity q_0. For the sake of completeness, this solution of Timoshenko beam theory, given in [11, pp. 197–204], is mentioned here:

$$w_0 = \frac{q_0 L^4}{24E\overline{I}} \left\{ \left[\left(\frac{x}{L}\right)^4 - \frac{3}{2}\left(\frac{x}{L}\right)^2 + \frac{5}{16} \right] - \frac{2(1+\nu)}{k} \left(\frac{h_0}{L}\right)^2 \left[\left(\frac{x}{L}\right)^2 - \frac{1}{4} \right] \right\}, \tag{37}$$

where $\overline{I} = bh_0^3/12$ and k is the shear-correction factor [1].

4. Numerical examples

To demonstrate the procedure followed in the present study, we present some numerical results for beams with uniform, linear, quadratic, and cubic thickness variations. In addition to the

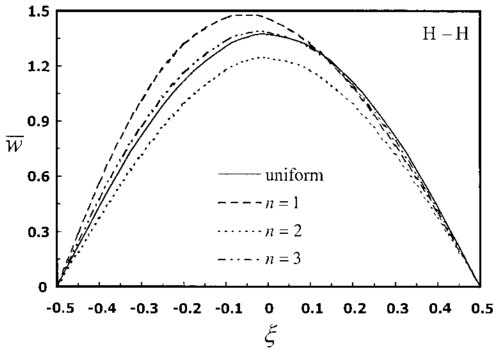

Figure 4. Distribution of the dimensionless deflection ($\overline{w}$) through the length of a hinged-hinged orthotropic beam for different thickness variations ($\lambda = 0{\cdot}3$, $L/h_0 = 10$).

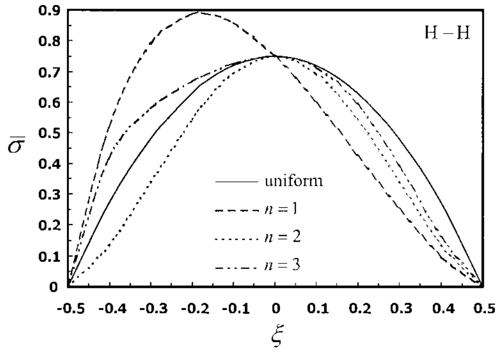

Figure 5. Distribution of the dimensionless bending stress ($\overline{\sigma}_1$) through the length of a hinged-hinged orthotropic beam for different thickness variations ($\lambda = 0{\cdot}3$).

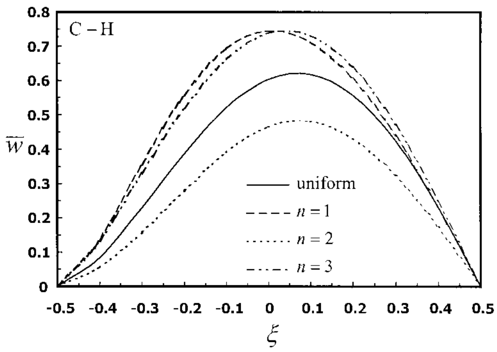

Figure 6. Distribution of the dimensionless deflection ($\overline{w}$) through the length of a clamped-hinged orthotropic beam for different thickness variations ($\lambda = 0{\cdot}3$, $L/h_0 = 10$).

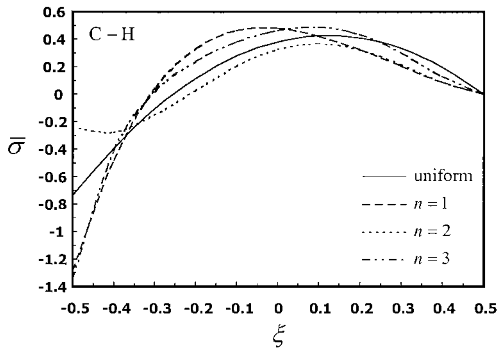

Figure 7. Distribution of the dimensionless bending stress ($\overline{\sigma_1}$) through the length of a clamped-hinged orthotropic beam for different thickness variations ($\lambda = 0{\cdot}3$).

isotropic material ($\nu = 0{\cdot}3$), the numerical calculations are carried out for an orthotropic material with the following dimensionless properties:

$$E_1 = 20{\cdot}83 \text{ Msi}, \quad G_{13} = 3{\cdot}71 \text{ Msi}, \quad \nu_{13} = 0{\cdot}44. \tag{38}$$

For the nonlinear thickness variation h the function $f_n(x)$ may be expressed as follows:

$$f_n(x) = \left(\frac{x}{L/2}\right)^n, \quad n = 1, 2, 3, \ldots \tag{39}$$

Equations (26) and (28) are solved separately and their contributions φ_m and $w_m(m = 0, 1, 2, \ldots)$ and the corresponding contributions w_0, w_1, w_2, and so on, to the total deflection $w = L(w_0 + \lambda w_1 + \lambda^2 w_2 + \lambda^3 w_3 + \ldots)$ of the beam are compared. The mid-span point ($x = 0$) and the different values of the thickness parameter λ are used for this purpose. Results are also obtained for combinations of the aspect ratio L/h_0 and the parame-

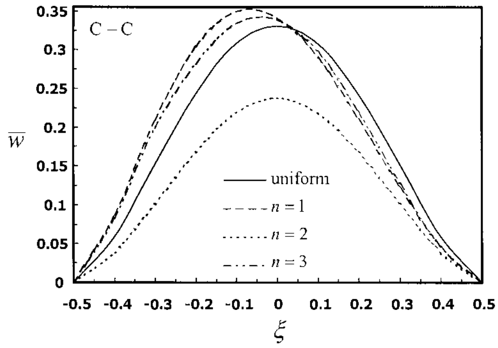

Figure 8. Distribution of the dimensionless deflection ($\overline{w}$) through the length of a clamped-clamped orthotropic beam for different thickness variations ($\lambda = 0{\cdot}3$, $L/h_0 = 10$).

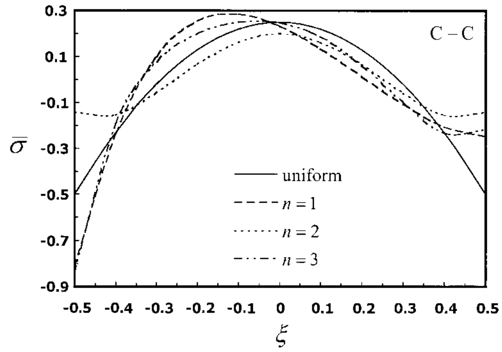

Figure 9. Distribution of the dimensionless bending stress ($\overline{\sigma_1}$) through the length of a clamped-clamped orthotropic beam for different thickness variations ($\lambda = 0{\cdot}3$).

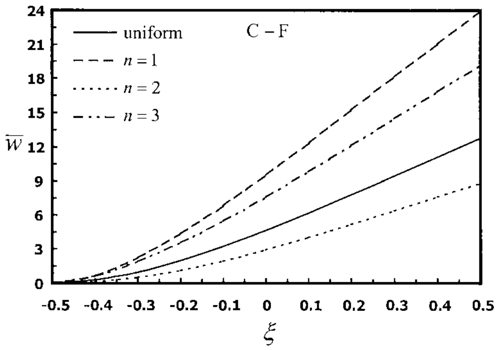

Figure 10. Distribution of the dimensionless deflection ($\overline{w}$) through the length of a clamped-free orthotropic beam for different thickness variations ($\lambda = 0{\cdot}3$, $L/h_0 = 10$).

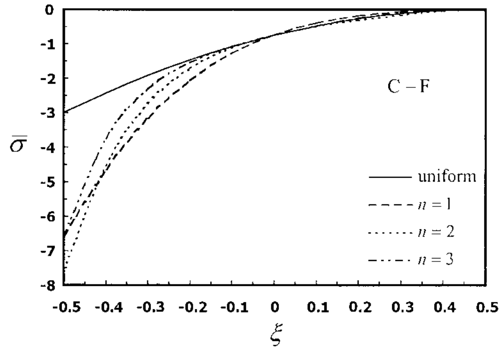

Figure 11. Distribution of the dimensionless normal stress ($\overline{\sigma_1}$) through the length of a clamped-free orthotropic beam for different thickness variations ($\lambda = 0{\cdot}3$).

ter $\xi(\equiv x/L)$. The deflections and normal stress being reported herein are of the following dimensionless forms:

$$\overline{w}_0 = \frac{10^2 c_{11}}{q_0 L^3} w_0, \quad \overline{w}_1 = \frac{10^3 c_{11}}{q_0 L^3} \lambda w_1, \quad \overline{w}_2 = \frac{10^4 c_{11}}{q_0 L^3} \lambda^2 w_2,$$

$$\overline{w}_3 = \frac{10^5 c_{11}}{q_0 L^3} \lambda^3 w_3, \quad \overline{w} = \frac{10^2 c_{11} w}{q_0 L^4}, \quad \overline{\sigma}_1 = -\frac{b}{q_0} \left(\frac{h_0}{L}\right)^2 \sigma_1(x, y, \eta).$$

The results of the present investigations are listed in Table 1 and Figures 2–15. Uniform distribution of loading in the spatial domain, $q(x) = q_0$, is used, where the domain of the beam is $-L/2 \leq x \leq L/2$. Note that the dimensionless stress $\overline{\sigma}_1$ is given in terms of the ratio L/h_0. Thus, the original bending stress $\sigma_1[\equiv -(\overline{\sigma}_1 q_0/b)\,(L/h_0)^2]$ is proportional to $(L/h_0)^2$.

The results are summarized in Table 1 for the center deflections of an isotropic beam with various thickness variations. The contributions $\overline{w}_0$, $\overline{w}_1$, $\overline{w}_2$, and $\overline{w}_3$ to the total deflection $\overline{w}$ of

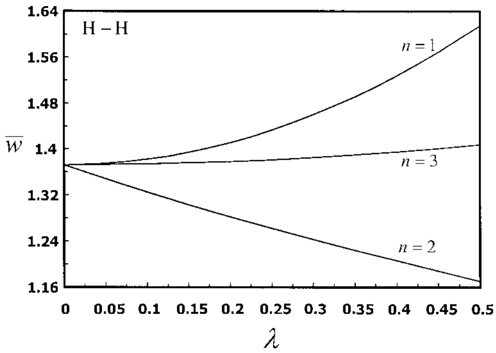

Figure 12. Dimensionless deflection ($\overline{w}$) *vs.* the thickness parameter (λ) of hinged-hinged orthotropic beam with linear, quadratic and cubit thickness variations ($\xi = 0$, $L/h_0 = 10$).

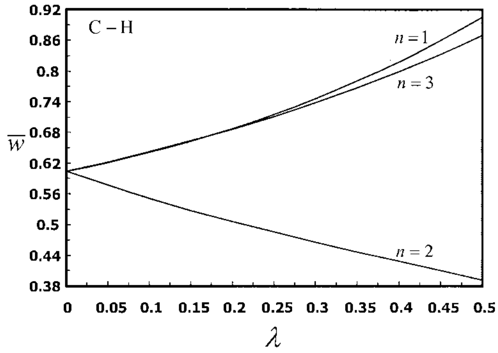

Figure 13. Dimensionless deflection ($\overline{w}$) *vs.* the thickness parameter (λ) of clamped-hinged orthotropic beam with linear, quadratic and cubit thickness variations ($\xi = 0$, $L/h_0 = 10$).

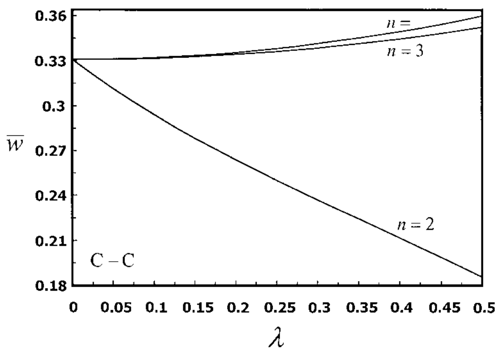

Figure 14. Dimensionless deflection ($\overline{w}$) *vs.* the thickness parameter (λ) of clamped-clamped orthotropic beam with linear, quadratic and cubit thickness variations ($\xi = 0$, $L/h_0 = 10$).

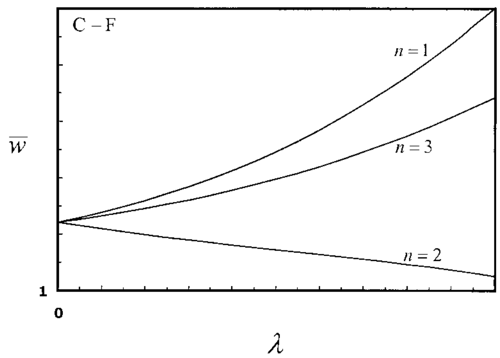

Figure 15. Dimensionless deflection ($\overline{w}$) *vs.* the thickness parameter (λ) of clamped-free orthotropic beam with linear, quadratic and cubit thickness variations ($\xi = 0$, $L/h_0 = 10$).

the beam for different values of the thickness parameter λ are compared in Table 1. Figures 2 and 3 display the variation of the dimensionless mid-span deflection $\overline{w}$ vs. the beam aspect ratio ($\equiv L/h_0$) for a hinged-hinged isotropic beam with linear ($n = 1$) and quadratic ($n = 2$) thickness variations, respectively. The results obtained within the present FBT are compared with their classical counterparts in Table 1 and Figures 2 and 3. Figures 4 and 5 display the variation of $\overline{w}$ and normal stress $\overline{\sigma}_1$ at the upper surface of the beam ($\eta = 0$) through the length of the hinged-hinged orthotropic beam with various thickness variations. Similar results for clamped-hinged, clamped-clamped, and clamped-free beams are given in Figures 6–11. Finally, the effect of the thickness parameter λ on the dimensionless mid-span deflection of beams with linear, quadratic, and cubic thickness variations subjected to various boundary conditions is shown in Figures 12–15. Note that the uniform beam deflections are, of course, given in these figures for $\lambda = 0$.

Table 1. Non-dimensional mid-span deflections of a hinged-hinged isotropic beam for different values of the thickness parameter $\lambda(L/h_0 = 10)$.

		$\overline{w}_0$		$\overline{w}_1$		$\overline{w}_2$		$\overline{w}_3$		$\overline{w}$	
n	λ	FBT	CBT	FBT	CBT	FBT	CBT	FBT	CBT	FBT	CBT
1	0·05	1·3346	1·3021	0	0	0·2384	0·2344	0	0	1·3370	1·3044
	0·10	1·3346	1·3021	0	0	0·9538	0·9375	0	0	1·3441	1·3115
	0·20	1·3346	1·3021	0	0	3·8150	3·7500	0	0	1·3727	1·3396
	0·30	1·346	1·3021	0	0	8·5838	8·4375	0	0	1·4204	1·3864
	0·40	1·3346	1·3021	0	0	15·2600	15·0000	0	0	1·4872	1·4521
2	0·05	1·3346	1·3021	−0·2425	−0·2344	0·0753	0·0725	−0·0274	−0·0264	1·3111	1·2793
	0·10	1·3346	1·3021	−0·4850	−0·4688	0·3010	0·2902	−0·2189	−0·2108	1·2889	1·2579
	0·20	1·3346	1·3021	−0·9700	−0·9375	·2040	1·1607	−1·7515	−1·6865	1·2479	1·2183
	0·30	1·3346	1·3021	−1·4550	−1·4063	2·7091	2·6116	−5·9113	−5·6920	1·2103	1·1819
	0·40	1·3346	1·3021	−1·9400	−1·8750	4·8162	4·6429	−14·0121	−13·4921	1·1747	1·1475
3	0·05	1·3346	1·3021	0	0	0·0.337	0·0316	0	0	1·3349	1·3024
	0·10	1·3346	1·3021	0	0	0·1346	0·1265	0	0	1·3359	1·3033
	0·20	1·3346	1·3021	0	0	0·5385	0·5060	0	0	1·3400	1·3071
	0·30	1·3346	1·3021	0	0	1·2115	1·1384	0	0	1·3467	1·3135
	0·40	1·3346	1·3021	0	0	2·1538	2·0238	0	0	1·3561	1·3223

5. Discussion

Examination of the results displayed in Table 1 and Figures 2 and 3, reveal that the deflections of a uniform beam $\overline{w}_0$ compare very well with those obtained using the exact solution given in the literature with any appropriate value of the shear-correction factor k. In fact, the present uniform-beam deflections are identical to those obtained using the exact solution given in (37) for the shear-correction factor $k = 5/6$. It is clear that the total deflections predicted by the FBT are higher than those of the CBT. This is due to the fact that the CBT represents beam behaviour as relatively more stiff. Increasing the values of the thickness parameter λ will lead to an increase of the total deflections for beams with linear and cubic thickness variations, while it will lead to a decrease of the total deflections for beams with quadratic thickness variation. Table 1 and Figures 2, 3 and 12–15 are very revealing in this respect.

Figure 4 reveals that the variation of $\overline{w}$ is very sensitive to variations of the parameter $\xi(\equiv x/L)$ of a hinged-hinged beam, irrespective of the considered thickness variations. In this sense, the linear thickness variation shows the highest sensitivity in the context of the considered thickness variations. Figure 5 reveals also the sensitivity of $\overline{\sigma}_1$ through-the-length of a hinged-hinged beam.

Concerning the influence played by the boundary conditions on the analyzed response characteristics, Figures 6–11 emphasize their great influence on the analyzed response characteristics.

6. Conclusions

The present FBT is a fairly simple theory that includes a shear-deformation contribution with what appears to be reasonably good accuracy. So, we may now be able to determine the normal stress σ_1 and the deflection curve $w(x)$ for short, as well as long beams. We have presented in this study only selected examples that illustrate the essential features of the present formulation. Exact solutions have been developed for orthotropic beams with various thickness variations, subjected to uniformly distributed loads. The effects of shear deformation, the thickness parameter λ, the aspect ratio L/h_0, the parameter $\xi(\equiv x/L)$, the thickness variation, and the boundary conditions on the deflection and normal stress have been investigated.

One of the goals of this paper has been to develop a simple theory for the bending of orthotropic beams, allowing one to incorporate the transverse-shear effect without using any shear-correction factor. By satisfying the transverse shear-stress-free conditions on the lateral surfaces of the beam, a pair of coupled equilibrium equations was obtained such that no arbitrary shear-correction factor is required. It has been shown that the uncoupled equation for the deflection is the same as the corresponding equation in Timoshenko's beam theory, provided that for Timoshenko's equation the shear-correction factor is taken as 5/6.

Another goal has been the use, in this context, of a powerful technique based on the small-parameter method allowing one to obtain exact solutions associated with the case of variable-thickness orthotropic beams. Additional results concerning the bending response of thin and thick plates with various thickness variations will be reported in future publications.

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